

# Improved Long-Period Generators Based on Linear Recurrences Modulo 2

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Fast uniform random number generators with extremely long periods have been defined and implemented based on linear recurrences modulo 2. The twisted GFSR and the Mersenne twister are famous recent examples. Besides the period length, the statistical quality of these generators is usually assessed via their equidistribution properties. The huge-period generators proposed so far are not quite optimal in that respect. In this paper, we propose new generators, with better equidistribution and “bit-mixing” properties for equivalent period length and speed. Approximately half of the coefficients of the characteristic polynomial of these generators are nonzero. The state of our new generators evolves in a more chaotic way than for the Mersenne twister. We illustrate how this can reduce the impact of persistent dependencies among successive output values, which can be observed in certain parts of the period of gigantic generators such as the Mersenne twister.

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## 1. INTRODUCTION

Most uniform random number generators (RNGs) used in computational statistics and simulation are based on linear recurrences modulo 2 or modulo a large integer. The main advantages of these generators are that fast implementations are available and that their mathematical properties can be studied in detail from a theoretical perspective. In particular, large-period linear RNGs are easy to design and the geometrical structure of the set  $\Psi_t$  of all vectors of  $t$  successive values produced by the generator, from all its possible initial states, can be precisely assessed without generating the points explicitly [L'Ecuyer 2004; Tezuka 1995]. In this paper, we are concerned with RNGs based on linear recurrences in  $\mathbb{F}_2$ , the finite field with two elements, 0 and 1, in which the arithmetic operations are equivalent to arithmetic modulo 2. Their output sequence is supposed to imitate i.i.d. random variables

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uniformly distributed over the real interval  $[0, 1]$ . Such recurrences are attractive because their implementation can exploit the binary nature of computers. The set  $\Psi_t$  in this context is a finite subset of the unit hypercube  $[0, 1]^t$  and it is customary to require that  $\Psi_t$  covers this unit hypercube very evenly for all  $t$  up to some large integer (as large as possible), because  $\Psi_t$  can be viewed as the “sample space” from which the  $t$ -dimensional points are drawn instead of drawing them uniformly from  $[0, 1]^t$  [L'Ecuyer 2004].

More specifically, we consider the matrix linear recurrence defined by

$$\mathbf{x}_i = \mathbf{A}\mathbf{x}_{i-1}, \quad (1)$$

$$\mathbf{y}_i = \mathbf{B}\mathbf{x}_i, \quad (2)$$

$$u_i = \sum_{\ell=1}^w y_{i,\ell-1} 2^{-\ell} = .y_{i,0} y_{i,1} y_{i,2} \cdots, \quad (3)$$

where  $\mathbf{x}_i = (x_{i,0}, \dots, x_{i,k-1})^\top \in \mathbb{F}_2^k$  and  $\mathbf{y}_i = (y_{i,0}, \dots, y_{i,w-1})^\top \in \mathbb{F}_2^w$  are the  $k$ -bit *state* and the  $w$ -bit *output vector* at step  $i$ ,  $\mathbf{A}$  and  $\mathbf{B}$  are a  $k \times k$  *transition matrix* and a  $w \times k$  *output transformation matrix* (both with elements in  $\mathbb{F}_2$ ),  $k$  and  $w$  are positive integers, and  $u_i \in [0, 1)$  is the *output* at step  $i$ . All operations in (1) and (2) are performed in  $\mathbb{F}_2$  and each element of  $\mathbb{F}_2$  is represented as one bit. In this setup, we have

$$\Psi_t = \{(u_0, u_1, \dots, u_{t-1}) : \mathbf{x}_0 \in \mathbb{F}_2^k\}.$$

By appropriate choices of  $\mathbf{A}$  and  $\mathbf{B}$ , several well-known types of generators can be obtained as special cases of this general class, including the Tausworthe, linear feedback shift register (LFSR), generalized feedback shift register (GFSR), twisted GFSR, Mersenne twister, and linear cellular automata [L'Ecuyer and Panneton 2002; Panneton 2004].

We briefly recall some (important) well-known facts about this class of RNGs. The *characteristic polynomial* of the matrix  $\mathbf{A}$  can be written as

$$P(z) = \det(\mathbf{A} - z\mathbf{I}) = z^k - \alpha_1 z^{k-1} - \cdots - \alpha_{k-1} z - \alpha_k,$$

where  $\mathbf{I}$  is the identity matrix and each  $\alpha_j$  is in  $\mathbb{F}_2$ . For each  $j$ , the sequences  $\{x_{i,j}, i \geq 0\}$  and  $\{y_{i,j}, i \geq 0\}$  both obey the linear recurrence

$$x_{i,j} = (\alpha_1 x_{i-1,j} + \cdots + \alpha_k x_{i-k,j}) \bmod 2 \quad (4)$$

[Niederreiter 1992; L'Ecuyer 1994]. (The output sequence  $\{y_{i,j}, i \geq 0\}$  could also obey a recurrence of shorter order in certain cases, depending on  $\mathbf{B}$ .) Here we assume that  $\alpha_k = 1$ , in which case (4) has *order*  $k$  and is purely periodic. The period length of the recurrence (4) has the upper bound  $2^k - 1$ , which is achieved if and only if  $P(z)$  is a primitive polynomial over  $\mathbb{F}_2$  [Niederreiter 1992; Knuth 1998]. In this paper, we are only interested in generators having this *maximal-period* property.

With this type of generator, to jump ahead directly from  $\mathbf{x}_i$  to  $\mathbf{x}_{i+\nu}$  for an arbitrary  $\nu$ , it suffices to precompute the matrix  $\mathbf{A}^\nu \bmod 2$  (i.e., in  $\mathbb{F}_2$ ) and then multiply  $\mathbf{x}_i$  by this matrix. This is useful for splitting the sequence into streams and substreams, e.g., as in L'Ecuyer et al. [2002].

In the next section of the paper, we recall basic definitions related to the (standard) use of equidistribution to measure the uniformity of  $\Psi_t$  for linear generators over  $\mathbb{F}_2$ . We also specify the uniformity measures adopted in this paper. In Section 3, we mention other linear generators over  $\mathbb{F}_2$  that are already available, including combined LFSRs [L’Ecuyer 1999b], GFSRs, the Mersenne twister [Matsumoto and Nishimura 1998; Nishimura 2000], and combined twisted GFSRs [L’Ecuyer and Panneton 2002], and point out drawbacks of some of them. In particular, some of the fastest generators make very few changes to the bits of the state  $\mathbf{x}_i$  from one step to the next. Then, if  $\mathbf{x}_i$  contains many more 0’s than 1’s, the same is likely to hold for  $\mathbf{x}_{i+1}$ . This suggests that to obtain good and robust generators, the matrices  $\mathbf{A}$  and  $\mathbf{B}$  in (1) and (2) must perform a large enough amount of bit transformations. Our aim in this paper is precisely to build new generators that perform more bit transformations and are better equidistributed than (for example) the Mersenne twister, while having the same period length and approximately the same speed. To achieve that, we construct a matrix  $\mathbf{A}$  that implements inexpensive operations such as bit shifts, xors, and bit masks, cleverly spread out in the matrix. For  $\mathbf{B}$  we simply use the identity. We prefer to put all the transformations in  $\mathbf{A}$  because these transformations carry over to the following states, whereas those in  $\mathbf{B}$  do not. Section 4 describes how we have designed our new generators. Section 5 gives specific instances found by a computer search, using REGPOLY [L’Ecuyer and Panneton 2002], for well-equidistributed generators having state spaces of various sizes. Section 6 deals with implementation issues and speed. In Section 7, we compare the TT800 generator of [Matsumoto and Kurita 1994] and the Mersenne twister of [Matsumoto and Nishimura 1998] with two of our new generators having the same period lengths ( $2^{800} - 1$  and  $2^{19937} - 1$ , respectively), in terms of their ability to quickly exit a region of the state space where the fraction of zeros in the state (or in the output vector  $\mathbf{y}_i$ ) is far away from  $1/2$ . The new generators perform much better in these experiments. That is, their behavior is more in line with randomness and chaos.

## 2. EQUIDISTRIBUTION AND MEASURES OF QUALITY

A convenient and rather standard way of measuring the uniformity of  $\Psi_t$  for linear RNGs over  $\mathbb{F}_2$  is as follows. Recall that  $\Psi_t$  has cardinality  $2^k$ . If we divide the interval  $[0, 1)$  into  $2^\ell$  equal segments for some positive integer  $\ell$ , this determines a partition of the unit hypercube  $[0, 1)^t$  into  $2^{t\ell}$  cubic cells of equal size, called a  $(t, \ell)$ -*equidissection* in base 2. The set  $\Psi_t$  is said to be  $(t, \ell)$ -*equidistributed* if each cell contains exactly  $2^{k-t\ell}$  of its points. Of course, this is possible only if

$$\ell \leq \ell_t^* \stackrel{\text{def}}{=} \min(w, \lfloor k/t \rfloor).$$

When the equidistribution holds for all pairs  $(t, \ell)$  that satisfy this condition, we say that  $\Psi_t$  (and the RNG) is *maximally-equidistributed* (ME). ME generators have the best possible equidistribution properties in terms of cubic equidissections. A major motivation for using this type of uniformity measure for linear generators over  $\mathbb{F}_2$  is that it can be efficiently computed in that case, without generating the points explicitly. The idea is simple: the first  $\ell$  bits of  $u_i, \dots, u_{i+t-1}$  form a  $t\ell$ -bit vector that can be written as  $\mathbf{M}_{t,\ell}\mathbf{x}_i$  for some  $t\ell \times k$  binary matrix  $\mathbf{M}_{t,\ell}$ , because

each output bit can be expressed as a linear combination of the bits of the current state, and the  $(t, \ell)$ -equidistribution holds if and only if the matrix  $\mathbf{M}_{t, \ell}$  has full rank [Fushimi and Tezuka 1983; L'Ecuyer 1996]. When  $k$  is very large, this matrix becomes expensive to handle, but in that case the equidistribution can be verified by a more efficient method based on the computation of the shortest nonzero vector in a lattice of formal series, as explained in Couture and L'Ecuyer [2000]. Large-period ME (or almost ME) generators have been proposed by L'Ecuyer [1999b], L'Ecuyer and Panneton [2002], and Panneton and L'Ecuyer [2004], for example. The aim of this paper is to propose new ones, with very large periods, and faster than those already available with comparable period lengths.

For non-ME generators, we denote  $t_\ell$  as the largest dimension  $t$  for which  $\Psi_t$  is  $(t, \ell)$ -equidistributed, and define the *dimension gap* for  $\ell$  bits of resolution as

$$\delta_\ell = t_\ell^* - t_\ell,$$

where  $t_\ell^* = \lfloor k/\ell \rfloor$  is an upper bound on the best possible value of  $t_\ell$ . As measures of uniformity, we consider the *worst-case dimension gap* and the *sum of dimension gaps*, defined as

$$\Delta_\infty = \max_{1 \leq \ell \leq w} \delta_\ell \quad \text{and} \quad \Delta_1 = \sum_{\ell=1}^w \delta_\ell.$$

Aside from equidistribution, it has been strongly advocated that good linear generators over  $\mathbb{F}_2$  must have characteristic polynomials  $P(z)$  whose number of nonzero coefficients is not too far from half the degree, i.e., in the vicinity of  $k/2$  [Compagner 1991; Wang and Compagner 1993]. In particular, generators for which  $P(z)$  is a trinomial or a pentanomial, which have been widely used in the past, do not satisfy this condition and have been shown to fail rather simple statistical tests [Lindholm 1968; Matsumoto and Kurita 1996]. So, as a secondary quality criterion, we look at the number of nonzero coefficients in  $P(z)$ , which we denote by  $N_1$ .

### 3. OTHER LINEAR GENERATORS OVER $\mathbb{F}_2$

A ME trinomial-based LFSR was proposed already by Tootill et al. [1973], who introduced the ME notion under the name of *asymptotically random*. Very fast algorithms are available for implementing trinomial-based LFSR generators whose parameters satisfy certain conditions, as well as for trinomial-based and pentanomial-based GFSR generators. However, these generators have much too few nonzero coefficients in their characteristic polynomials ( $N_1$  is 3 or 5, which is too small). One way of getting fast generators with a large value of  $N_1$ , proposed by Tezuka and L'Ecuyer [1991] and Wang and Compagner [1993] and pursued by L'Ecuyer [1996; 1999b], is to combine several trinomial-based LFSR generators of relatively prime period lengths, by bitwise xor. This gives another LFSR whose characteristic polynomial  $P(z)$  is the product of the characteristic polynomials of the components, so it may contain many more nonzero coefficients, e.g., up to  $3^J$  if we combine  $J$  trinomial-based LFSRs and  $3^J < k$ . The polynomial  $P(z)$  cannot be primitive in this context, but the period length can nevertheless be very close to  $2^k$  where  $k$  is the degree of  $P(z)$ . This method is quite effective to build generators with values of  $N_1$  up to a few hundreds on 32-bit computers, but for values in the thousands or

tens of thousands, one needs a large number  $J$  of components and this makes the implementation slower.

The twisted GFSR and Mersenne twister [Matsumoto and Kurita 1994; Matsumoto and Nishimura 1998; Nishimura 2000] provide very efficient implementations of linear generators over  $\mathbb{F}_2$  with primitive characteristic polynomials of very large degree  $k$ . However, their values of  $N_1$  are typically much smaller than  $k/2$  and they often have a large value of  $\Delta_1$ , due to the fact that their equidistribution is far from optimal in large dimensions. For example, MT19937 has  $k = 19937$ ,  $N_1 = 135$ , and  $\Delta_1 = 6750$ . This can be improved by combining several twisted GFSRs or Mersenne twisters as in L'Ecuyer and Panneton [2002], but at the expense of getting a slower generator.

Our goal in this paper was to build  $\mathbb{F}_2$ -linear generators with primitive characteristic polynomials, with speed and period length comparable to the Mersenne twister, and for which  $N_1 \approx k/2$  and  $\Delta_1 = 0$  (or nearly). We attach the acronym WELL, for “Well Equidistributed Long-period Linear”, to these new generators.

#### 4. STRUCTURE OF THE NEW PROPOSED GENERATORS

The general recurrence for the WELL generators is defined by the algorithm given in Figure 1, which uses the following notation. We decompose  $k$  as  $k = rw - p$  where  $r$  and  $p$  are the unique integers such that  $r > 0$  and  $0 \leq p < w$ . The state vector  $\mathbf{x}_i$  is decomposed into  $w$ -bit blocks as  $\mathbf{x}_i = (\mathbf{v}_{i,0}, \dots, \mathbf{v}_{i,r-1})$  where the last  $p$  bits of  $\mathbf{v}_{i,r-1}$  are always zero, and  $\mathbf{T}_0, \dots, \mathbf{T}_7$  are  $w \times w$  binary matrices that apply linear transformations to these  $w$ -bit blocks. The integers  $k, p, m_1, m_2, m_3$ , where  $0 < m_1, m_2, m_3 < r$ , and the matrices  $\mathbf{T}_0, \dots, \mathbf{T}_7$ , are chosen so that  $P(z)$ , the characteristic polynomial of  $\mathbf{A}$ , with degree  $k = rw - p$ , is primitive over  $\mathbb{F}_2$ . We denote by  $\text{rot}_p$  a left rotation by  $p$  bits,  $\mathbf{m}_p$  is a bit mask that keeps the first  $w - p$  bits and sets all other bits to zero and “ $\oplus$ ” and “ $\leftarrow$ ” represents the bitwise xor and the assignment statement, respectively. The temporary variables  $\mathbf{z}_0, \dots, \mathbf{z}_4$  are  $w$ -bit vectors.

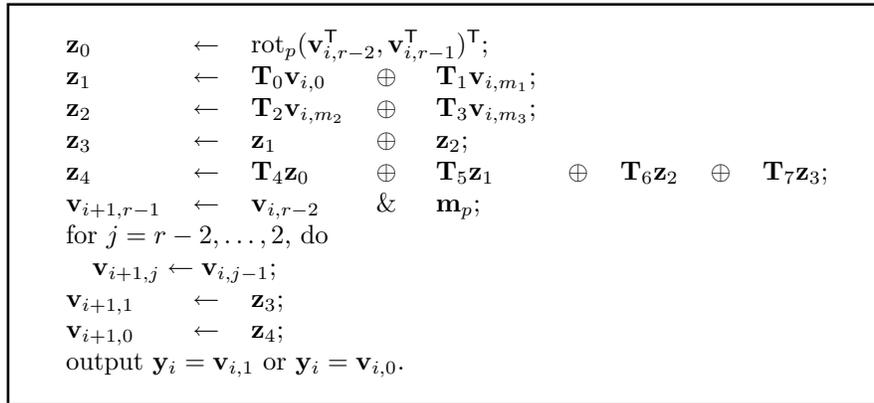


Fig. 1. The WELL algorithm

Table I. Possibilities for the transformations  $\mathbf{T}_j$ 

Transformation matrix $\mathbf{M}$	Implementation of $\mathbf{y} = \mathbf{M}\mathbf{x}$
$\mathbf{M}_0$	$\mathbf{y} = 0$
$\mathbf{M}_1$	$\mathbf{y} = \mathbf{x}$
$\mathbf{M}_2(t)$ $-w \leq t \leq w$	$\mathbf{y} = \begin{cases} (\mathbf{x} \gg t) & \text{if } t \geq 0 \\ (\mathbf{x} \ll -t) & \text{otherwise} \end{cases}$
$\mathbf{M}_3(t)$ $-w \leq t \leq w$	$\mathbf{y} = \begin{cases} \mathbf{x} \oplus (\mathbf{x} \gg t) & \text{if } t \geq 0 \\ \mathbf{x} \oplus (\mathbf{x} \ll -t) & \text{otherwise} \end{cases}$
$\mathbf{M}_4(\mathbf{a})$ $\mathbf{a} \in \mathbb{F}_2^w$	$\mathbf{y} = \begin{cases} (\mathbf{x} \gg 1) \oplus \mathbf{a} & \text{if } x^{(w-1)} = 1 \\ (\mathbf{x} \gg 1) & \text{otherwise} \end{cases}$
$\mathbf{M}_5(t, \mathbf{b})$ $-w \leq t \leq w$ $\mathbf{b} \in \mathbb{F}_2^w$	$\mathbf{y} = \begin{cases} \mathbf{x} \oplus ((\mathbf{x} \ll t) \& \mathbf{b}) & \text{if } t \geq 0 \\ \mathbf{x} \oplus ((\mathbf{x} \gg -t) \& \mathbf{b}) & \text{otherwise} \end{cases}$
$\mathbf{M}_6(r, s, t, \mathbf{a})$ $0 \leq r, s, t < w$ $\mathbf{a} \in \mathbb{F}_2^w$	$\mathbf{y} = \begin{cases} (((\mathbf{x} \ll r) \oplus (\mathbf{x} \gg (w-r))) \& \mathbf{d}_s) \oplus \mathbf{a} & \text{if } x^{(t)} = 1 \\ (((\mathbf{x} \ll r) \oplus (\mathbf{x} \gg (w-r))) \& \mathbf{d}_s) \oplus \mathbf{a} & \text{otherwise} \end{cases}$

Let  $\mathbf{T}_{i,j,k} = \mathbf{T}_i \mathbf{T}_k \oplus \mathbf{T}_j \mathbf{T}_k$  and  $\mathbf{U}_p$  and  $\mathbf{L}_{w-p}$  be the  $w \times w$  matrices such that

$$\mathbf{U}_p = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I}_p \end{pmatrix} \quad \text{and} \quad \mathbf{L}_{w-p} = \begin{pmatrix} \mathbf{I}_{w-p} & 0 \\ 0 & 0 \end{pmatrix}$$

where  $\mathbf{I}_p$  and  $\mathbf{I}_{w-p}$  are the identity matrices of size  $p$  and  $w-p$ , respectively. The matrix  $\mathbf{A}$  that corresponds to the WELL algorithm has the following block structure, with six  $w \times w$  nonzero submatrices on the first line and four on the second line:

$$\begin{pmatrix} \mathbf{T}_{5,7,0} & 0 & 0 & \dots & \mathbf{T}_{5,7,1} & 0 & \dots & \mathbf{T}_{6,7,2} & 0 & \dots & \mathbf{T}_{6,7,3} & \dots & \mathbf{T}_4 \mathbf{U}_p & \mathbf{T}_4 \mathbf{L}_{w-p} \\ \mathbf{T}_0 & 0 & 0 & \dots & \mathbf{T}_1 & 0 & \dots & \mathbf{T}_2 & 0 & \dots & \mathbf{T}_3 & \dots & 0 & 0 \\ 0 & \mathbf{I} & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ 0 & 0 & \mathbf{I} & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & 0 \\ & & & & & & \ddots & & & & & & & \\ 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 & \dots & \mathbf{I} & 0 \end{pmatrix}$$

If both  $\mathbf{T}_0$  and  $\mathbf{T}_4$  have full rank, then  $\mathbf{A}$  has rank  $wr - p$ .

The transformations (matrices)  $\mathbf{T}_j$  are selected among the possibilities enumerated in Table I. In this table, the bit vector  $\mathbf{d}_s$  has its  $(s+1)$ -th bit set to zero and all the others are set to one. These transformations can be implemented very efficiently by just a few operations on a  $w$ -bit block.

## 5. SPECIFIC PARAMETERS

Table II lists specific parameters for generators of period length ranging from  $2^{512} - 1$  to  $2^{44497} - 1$ . The values of the vectors  $\mathbf{a}_i$  are  $\mathbf{a}_1 = \text{da442d24}$ ,  $\mathbf{a}_2 = \text{d3e43ffd}$ ,  $\mathbf{a}_3 = \text{8bdc91e}$ ,  $\mathbf{a}_4 = \text{86a9d87e}$ ,  $\mathbf{a}_5 = \text{a8c296d1}$ ,  $\mathbf{a}_6 = \text{5d6b45cc}$  and  $\mathbf{a}_7 = \text{b729fcec}$ . Several of the proposed generators are ME (they have  $\Delta_1 = 0$ ). For example, all the generators listed with  $k = 512, 521, 607, 1024$  are ME. We did not find ME generators with  $\mathbf{B} = \mathbf{I}$  for  $k = 800, 19937, 21701, 23209, 44497$ , but all those given in Table II have a very small value of  $\Delta_1$  (from 1 to 7) and values of  $N_1$  not far from  $k/2$ . For comparison, the TT800 generator proposed by Matsumoto and Kurita [1994], with period length  $2^{800} - 1$ , has  $\Delta_1 = 261$  and  $N_1 = 93$ , whereas the Mersenne twister MT19937 of Matsumoto and Nishimura [1998] has  $\Delta_1 = 6750$  and

Table II. Specific well-equidistributed generators

$m_1$	$m_2$	$m_3$	$\mathbf{T}_0$ $\mathbf{T}_4$	$\mathbf{T}_1$ $\mathbf{T}_5$	$\mathbf{T}_2$ $\mathbf{T}_6$	$\mathbf{T}_3$ $\mathbf{T}_7$	$\Delta_1$ $N_1$
$k = 512, w = 32, r = 16, p = 0$							
			$\mathbf{M}_3(-16)$	$\mathbf{M}_3(-15)$	$\mathbf{M}_3(11)$	$\mathbf{M}_0$	0
13	9	5	$\mathbf{M}_3(-2)$	$\mathbf{M}_3(-18)$	$\mathbf{M}_3(-28)$	$\mathbf{M}_5(-5, \mathbf{a}_1)$	225
$k = 521, w = 32, r = 17, p = 23$							
			$\mathbf{M}_3(-13)$	$\mathbf{M}_3(-15)$	$\mathbf{M}_1$	$\mathbf{M}_2(-21)$	0
13	11	10	$\mathbf{M}_3(-13)$	$\mathbf{M}_2(1)$	$\mathbf{M}_0$	$\mathbf{M}_3(11)$	265
			$\mathbf{M}_3(-21)$	$\mathbf{M}_3(6)$	$\mathbf{M}_0$	$\mathbf{M}_3(-13)$	0
11	10	7	$\mathbf{M}_3(13)$	$\mathbf{M}_2(-10)$	$\mathbf{M}_2(-5)$	$\mathbf{M}_3(13)$	245
$k = 607, w = 32, r = 19, p = 1$							
			$\mathbf{M}_3(19)$	$\mathbf{M}_3(11)$	$\mathbf{M}_3(-14)$	$\mathbf{M}_1$	0
16	15	14	$\mathbf{M}_3(18)$	$\mathbf{M}_1$	$\mathbf{M}_0$	$\mathbf{M}_3(-5)$	295
			$\mathbf{M}_3(-18)$	$\mathbf{M}_3(-14)$	$\mathbf{M}_0$	$\mathbf{M}_3(18)$	0
16	8	13	$\mathbf{M}_3(-24)$	$\mathbf{M}_3(5)$	$\mathbf{M}_3(-1)$	$\mathbf{M}_0$	313
$k = 800, w = 32, r = 25, p = 0$							
			$\mathbf{M}_1$	$\mathbf{M}_3(-15)$	$\mathbf{M}_3(10)$	$\mathbf{M}_3(-11)$	3
14	18	17	$\mathbf{M}_3(16)$	$\mathbf{M}_2(20)$	$\mathbf{M}_1$	$\mathbf{M}_3(-28)$	303
			$\mathbf{M}_3(-29)$	$\mathbf{M}_2(-14)$	$\mathbf{M}_1$	$\mathbf{M}_2(19)$	3
9	4	22	$\mathbf{M}_1$	$\mathbf{M}_3(10)$	$\mathbf{M}_4(\mathbf{a}_2)$	$\mathbf{M}_3(-25)$	409
$k = 1024, w = 32, r = 32, p = 0$							
			$\mathbf{M}_1$	$\mathbf{M}_3(8)$	$\mathbf{M}_3(-19)$	$\mathbf{M}_3(-14)$	0
3	24	10	$\mathbf{M}_3(-11)$	$\mathbf{M}_3(-7)$	$\mathbf{M}_3(-13)$	$\mathbf{M}_0$	407
			$\mathbf{M}_3(-21)$	$\mathbf{M}_3(17)$	$\mathbf{M}_4(\mathbf{a}_3)$	$\mathbf{M}_3(15)$	0
22	25	26	$\mathbf{M}_3(-14)$	$\mathbf{M}_3(-21)$	$\mathbf{M}_1$	$\mathbf{M}_0$	475
$k = 19937, w = 32, r = 624, p = 31$							
			$\mathbf{M}_3(-25)$	$\mathbf{M}_3(27)$	$\mathbf{M}_2(9)$	$\mathbf{M}_3(1)$	4
70	179	449	$\mathbf{M}_1$	$\mathbf{M}_3(-9)$	$\mathbf{M}_3(-21)$	$\mathbf{M}_3(21)$	8585
			$\mathbf{M}_3(7)$	$\mathbf{M}_1$	$\mathbf{M}_3(12)$	$\mathbf{M}_3(-10)$	5
203	613	123	$\mathbf{M}_3(-19)$	$\mathbf{M}_2(-11)$	$\mathbf{M}_3(4)$	$\mathbf{M}_3(-10)$	9679
			WELL19937a with M-K tempering				0
			$\mathbf{b}=\mathbf{e46e1700}, \mathbf{c}=\mathbf{9b868000}$				8585
$k = 21701, w = 32, r = 679, p = 27$							
			$\mathbf{M}_1$	$\mathbf{M}_3(-26)$	$\mathbf{M}_3(19)$	$\mathbf{M}_0$	1
151	327	84	$\mathbf{M}_3(27)$	$\mathbf{M}_3(-11)$	$\mathbf{M}_6(15, 10, 27, \mathbf{a}_4)$	$\mathbf{M}_3(-16)$	7609
$k = 23209, w = 32, r = 726, p = 23$							
			$\mathbf{M}_3(28)$	$\mathbf{M}_1$	$\mathbf{M}_3(18)$	$\mathbf{M}_3(3)$	3
667	43	462	$\mathbf{M}_3(21)$	$\mathbf{M}_3(-17)$	$\mathbf{M}_3(-28)$	$\mathbf{M}_3(-1)$	10871
			$\mathbf{M}_4(\mathbf{a}_5)$	$\mathbf{M}_1$	$\mathbf{M}_6(15, 30, 15, \mathbf{a}_6)$	$\mathbf{M}_3(-24)$	3
610	175	662	$\mathbf{M}_3(-26)$	$\mathbf{M}_1$	$\mathbf{M}_0$	$\mathbf{M}_3(16)$	10651
$k = 44497, w = 32, r = 1391, p = 15$							
			$\mathbf{M}_3(-24)$	$\mathbf{M}_3(30)$	$\mathbf{M}_3(-10)$	$\mathbf{M}_2(-26)$	7
23	481	229	$\mathbf{M}_1$	$\mathbf{M}_3(20)$	$\mathbf{M}_6(9, 14, 5, \mathbf{a}_7)$	$\mathbf{M}_1$	16883
			WELL44497a with M-K tempering				0
			$\mathbf{b}=\mathbf{93dd1400}, \mathbf{c}=\mathbf{fa118000}$				16883

$N_1 = 135$ . Table III complements Table II by giving, for each generator, the values of  $\ell$  for which  $\delta_\ell > 0$ .

For the generators that are not ME, it is easy to make them ME by adding a Matsumoto-Kurita tempering to the output. This is done with the following

Table III. Nonzero dimension gaps for the proposed WELL's

WELL	$\{\ell : \delta_\ell = 1\}$	$\{\ell : \delta_\ell > 1\}$	$\Delta_1$
800a	{20, 25, 32}	$\emptyset$	3
800b	{5, 17, 25}	$\emptyset$	3
19937a	{2, 7, 15, 28}	$\emptyset$	4
19937b	{3, 9, 14, 16, 32}	$\emptyset$	5
21701a	{20}	$\emptyset$	1
23209a	{6, 23, 24}	$\emptyset$	3
23209b	{3, 4, 12}	$\emptyset$	3
44497a	{2, 3, 4, 8, 16, 24, 27}	$\emptyset$	7

operations:

$$\begin{aligned} \mathbf{z} &\leftarrow \text{trunc}_w(\mathbf{x}_n) \\ \mathbf{z} &\leftarrow \mathbf{z} \oplus ((\mathbf{z} \ll 7) \& \mathbf{b}) \\ \mathbf{y}_n &\leftarrow \mathbf{z} \oplus ((\mathbf{z} \ll 15) \& \mathbf{c}) \end{aligned}$$

where  $\mathbf{b}$  and  $\mathbf{c}$  are carefully selected  $w$ -bit vectors and the operator  $\text{trunc}_w$  keeps only the first  $w$  bits. The output value  $u_n$  is generated via equation (3) with this  $\mathbf{y}_n$ . Adding this tempering to WELL44497a with  $\mathbf{b} = \mathbf{93dd1400}$  and  $\mathbf{c} = \mathbf{fa118000}$  provides the largest ME generator with 32 bits of accuracy known so far. Adding this tempering to WELL19937a with  $\mathbf{b} = \mathbf{e46e1700}$  and  $\mathbf{c} = \mathbf{9b868000}$  transforms it into a ME generator.

To find the generators listed in Table II, we did a random search using REGPOLY [L'Ecuyer and Panneton 2002]. It was designed to look for full period generators that use a low number of binary operations. For example, it would not consider the case where  $\mathbf{T}_0 = \dots = \mathbf{T}_7 = \mathbf{M}_6$  because  $\mathbf{M}_6$  is the most expensive of the matrices of Table I. Once a full-period generator is found, its equidistribution properties are verified.

To find a full period generator, we must find a transition matrix having a primitive characteristic polynomial. To do this, we find the minimal polynomial over  $\mathbb{F}_2$  of the sequence  $\{y_{n,0}\}_{n \geq 0}$  generated by the first bit of the  $\mathbf{y}_n$ 's. If this polynomial is of degree  $k$ , then it is the characteristic polynomial of the transition matrix, otherwise it is a divisor of it. In the former case, we test this polynomial for primitivity using an algorithm proposed by Rieke et al. [1998] if  $2^k - 1$  is not prime. If  $2^k - 1$  is a Mersenne prime, then we can switch to an irreducibility test which is simpler and equivalent. For this purpose, we used a combination of the sieving algorithm described by Brent et al. [2003] and the Berlekamp [1970] algorithm implemented in the software package ZEN [Chabaud and Lercier 2000]. The method used to compute the equidistribution is the one introduced by Couture and L'Ecuyer [2000].

## 6. IMPLEMENTATION AND PERFORMANCE

Figure 2 gives an implementation in C of WELL1024a, whose parameters are given in Table II. This implementation uses an array of  $r = 32$  `unsigned int`'s to store the  $\mathbf{v}_{n,i}$  vectors. The integer `state_n` is equal to  $n \bmod r$ . Since  $r$  is a power of 2, the operation “modulo  $2^r$ ” can be implemented efficiently by a bit mask. In our case, the bit mask is `0x0000001f`. The vector  $\mathbf{v}_{n,i}$  is stored in `STATE[(r - n + i) mod r]`. By decrementing `state_n` at each function call, we

```

#define R 32
#define M1 3
#define M2 24
#define M3 10
#define MAT3POS(t,v) (v^(v>>t))
#define MAT3NEG(t,v) (v^(v<<(-t)))
#define Identity(v) (v)
#define V0      STATE[ state_n          ]
#define VM1     STATE[ (state_n+M1) & 0x0000001fUL ]
#define VM2     STATE[ (state_n+M2) & 0x0000001fUL ]
#define VM3     STATE[ (state_n+M3) & 0x0000001fUL ]
#define VRm1    STATE[ (state_n+31) & 0x0000001fUL ]
#define newV0   STATE[ (state_n+31) & 0x0000001fUL ]
#define newV1   STATE[ state_n          ]
static unsigned int z0, z1, z2, state_n;
static unsigned int STATE[R];

void InitWELL1024a (unsigned int *init){
    int j;
    state_n = 0;
    for (j = 0; j < R; j++) STATE[j] = init[j];
}

double WELL1024a (void){
    z0 = VRm1;
    z1 = Identity(V0) ^ MAT3POS (8, VM1);
    z2 = MAT3NEG (-19, VM2) ^ MAT3NEG(-14,VM3);
    newV1 = z1 ^ z2;
    newV0 = MAT3NEG (-11,z0) ^ MAT3NEG(-7,z1) ^ MAT3NEG(-13,z2) ;
    state_n = (state_n + 31) & 0x0000001fUL;
    return ((double) STATE[state_n] * 2.32830643653869628906e-10);
}

```

Fig. 2. A C implementation of WELL1024a

automatically perform the “for” loop of the generator algorithm, saving costly assignment operations.

Table IV. Time to generate and sum  $10^9$  random numbers

Generator	Time (s)
WELL512a	36.0
WELL1024a	36.0
WELL19937a	36.9
WELL44497a	41.3
F2wLFSR2_31_800	39.5
F2wLFSR3_7_800	32.8
TT800	44.0
MT19937	31.6
MRG32k3a	101

In table IV, we give the CPU time taken by a program that produces  $10^9$  random numbers and sums them, using different generators. The clocked generators are WELL1024a, WELL512a, WELL19937a, WELL44497a, the Mersenne Twister [Matsumoto and Nishimura 1998], TT800 [Matsumoto and Kurita 1994] F2wLFSR3\_7\_800, and F2wLFSR2\_31\_800 [Panneton and L’Ecuyer 2004], which

are all fast generators based on linear recurrences modulo 2. For comparison, we also include the widely used MRG32k3a [L'Ecuyer 1999a]. Our implementations of WELL512a, WELL19937a, and WELL44497a can be found on the web page of the second author and, for the other generators, the implementations used are the ones published in the original papers. The test was performed on a computer equipped with a 2.8Ghz Intel Pentium 4 processor and a Linux operating system, and the program was compiled using `gcc` with the `-O2` optimization flag.

The WELL512a, WELL1024a, WELL19937a and WELL44497a passed with success all the statistical tests included in the batteries Smallcrush, Crush and Bigcrush of TestU01 [L'Ecuyer and Simard 2002], except those that look for linear dependencies in a long sequence of bits, such as the matrix-rank test [Marsaglia 1985] for very large binary matrices and the linear complexity tests [Erdmann 1992]. This is in fact a limitation of *all*  $\mathbb{F}_2$ -linear generators, including the Mersenne twister, the TT800, and many others: because of their linear nature, the sequences produced by these generators just cannot have the linear complexity of a truly random sequence. This is definitely unacceptable in cryptology, for example, but is quite acceptable for the vast majority of simulation applications if the linear dependencies are of long range and high order.

## 7. ESCAPING ZEROLAND

In this section, we examine how quickly the WELL generators can escape from a bad initialization, e.g., one that contains only a few bits set to 1. We compare them with other generators, for which the initialization often has a long-lasting impact in the sense that the fraction of bits set to 1 in both the state and the output remains small for a large number of steps if the initial number is very small.

Our experiments are based on the following framework. Let  $S_1 = \{\mathbf{e}_j : 1 \leq j \leq k\}$ , where  $\mathbf{e}_j$  denotes the  $j$ -th unit vector. This is the set of states for which a single bit is set to 1. Let  $\mathbf{y}_i^{(j)}$  be the output vector  $\mathbf{y}_i$  at step  $i$  when the initial state is  $\mathbf{x}_0 = \mathbf{e}_j$ , and let  $u_i^{(j)}$  be the corresponding output value  $u_i$ . We denote by  $H(\mathbf{x})$  the *Hamming weight* of a bit vector  $\mathbf{x}$ , i.e., the number of its bits that are set to 1, and define

$$\gamma_{n,p} = \frac{1}{pkw} \sum_{i=n}^{n+p-1} \sum_{j=1}^k H(\mathbf{y}_i^{(j)}). \quad (5)$$

This moving average represents the fraction of bits that are 1 in the  $p$  successive output vectors  $\mathbf{y}_n, \dots, \mathbf{y}_{n+p-1}$ , averaged over all initial seeds in  $S_1$ . Under the null hypothesis that the  $\mathbf{y}_n$ 's are independent bit vectors uniformly distributed over  $\mathbb{F}_2^w$ ,  $\gamma_{n,p}$  should be approximately normally distributed with mean  $1/2$  and variance  $1/(4pkw)$ .

Figure 3 illustrates the behavior of  $\gamma_{n,100}$  for  $n = 100, \dots, 10^5$ , for the WELL800a and TT800 generators. We see that WELL800a reaches the neighborhood of 0.5 much faster than TT800. For the latter,  $\gamma_{n,100}$  is often below 0.49 for  $n$  up to more than 60000. Here, the standard deviation of  $\gamma_{n,100}$  should be  $1/3200$ , so 0.49 is 32 standard deviations away from the mean.

In Figure 4, we see a similar comparison between WELL19937a and MT19937, with  $\gamma_{n,1000}$  and  $n$  up to one million. Again, the WELL generator is much better

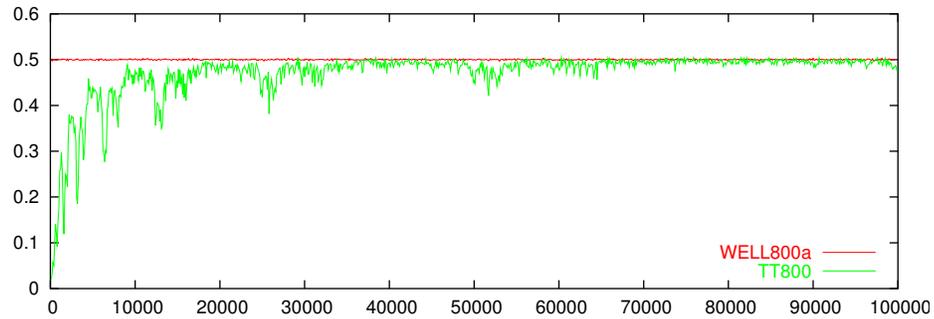


Fig. 3.  $\gamma_{n,100}$  for the WELL800a (top line) and TT800 (lower line)

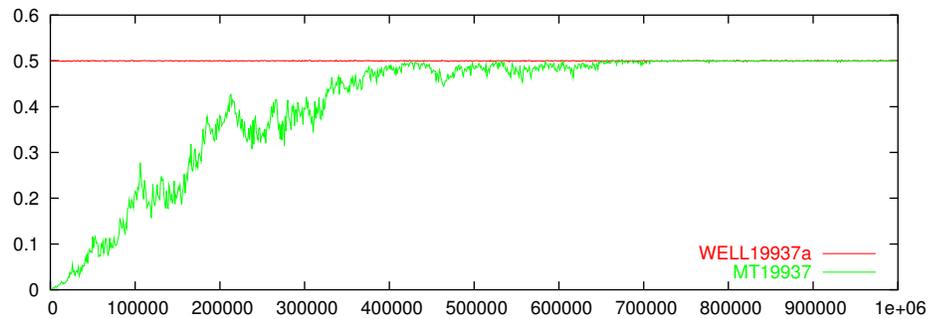


Fig. 4.  $\gamma_{n,1000}$  for the WELL19937a (top line) and MT19937 (lower line)

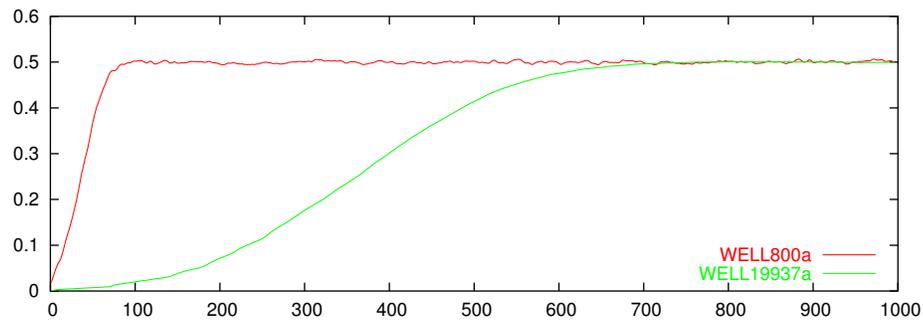


Fig. 5.  $\gamma_{n,5}$  for the WELL800a (top line) and WELL19937a (lower line)

behaved. With MT19937,  $\gamma_{n,1000}$  has an easily visible negative bias for up to at least  $n = 700000$  steps.

Although not visible in the scale of these two figures, the WELL generators also need a certain number of steps to bring their states close to the “center” of the state space, where roughly half of the bits are 1’s. To show that, Figure 5 gives a close-up view of the early behavior of  $\gamma_{n,5}$  for WELL800a and WELL19937a. There is still a visible (and significant) amount of dependence from the initial state, but the number of steps required to clear up the effect is approximately 1000 times

smaller than for TT800 and MT19937.

We also computed similar figures but with  $u_i^{(j)}$  instead of  $H(\mathbf{y}_i^{(j)})/w$  in (5), i.e., the average output value instead of the fraction of bits set to 1, and the results were almost identical. That alternative gives more weight to the more significant bits.

The behavior of the Hamming weight that we just examined is closely related to the *diffusion capacity* of generators. Crudely speaking, a generator is said to have good diffusion capacity if it produces two sequences that are very different from each other when started from two seeds that are very close to each other [Shannon 1949]. A precise definition depends on how we measure the closeness of seeds and the differences of sequences.

For the generators considered in this paper, we may define the distance between two seeds or two output vectors as the number of bits that differ between them. This is the *Hamming distance*. If two seeds  $\mathbf{x}'_0$  and  $\mathbf{x}''_0$  are at distance 1, then  $\mathbf{x}'_0 \oplus \mathbf{x}''_0 = \mathbf{e}_j$  for some  $j$ . If  $\mathbf{y}'_i$  and  $\mathbf{y}''_i$  are the corresponding output vectors at step  $i$ , then because of the linearity, we have  $\mathbf{y}'_i \oplus \mathbf{y}''_i = \mathbf{BA}^i(\mathbf{x}'_0 \oplus \mathbf{x}''_0) = \mathbf{BA}^i\mathbf{e}_j = \mathbf{y}_i^{(j)}$ , so the distance between  $\mathbf{y}'_i$  and  $\mathbf{y}''_i$  is the Hamming weight of  $\mathbf{y}_i^{(j)}$ . In this sense, the average Hamming weight defined in (5) also measures the diffusion capacity of the generator.

## 8. CONCLUSION

Random number generators based on linear recurrences modulo 2 are popular and very convenient for simulation, mainly because of their high speed and because the quality of specific instances can be studied in detail from the mathematical viewpoint. Good generators from this family have been proposed in the past (for instance, the Mersenne twister and the combined LFSRs). Some of them have huge period lengths, so their states must be represented over a large number of bits, and require only a few operations on 32-bit (or 64-bit) words to go from one state to the next. This means that only a small fraction of the bits in the state are modified at each step. In this paper, we have proposed a class of generators that do better than the previous ones on this aspect, without compromising the period length and speed. The improvement was assessed by a measure of equidistribution, by counting the fraction of nonzero coefficients in the characteristic polynomial of the recurrence, and by an empirical comparison of the diffusion capacity of the old and new generators. Of course, like all linear generators over  $\mathbb{F}_2$ , these generators cannot pass a statistical test that measures the linear complexity of a long sequence of bits that they produce. But they are nevertheless very useful and safe for the vast majority of simulation applications, especially when the generator's speed is important.

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